

Communities and classes in symmetric fractals

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Abstract

Two aspects of fractal networks are considered: the community structure and the class structure, where classes of nodes appear as a consequence of a local symmetry of nodes. The analysed systems are the networks constructed for two selected symmetric fractals: the Sierpinski triangle and the Koch curve. Communities are searched for by means of a set of differential equations. Overlapping nodes which belong to two different communities are identified by adding some noise to the initial connectivity matrix. Then, a node can be characterized by a spectrum of probabilities of belonging to different communities. Our main goal is that the overlapping nodes with the same spectra belong to the same class.

1 Introduction

Analysis of properties of networks may regard different aspects. One of them is the identification of communities; it is equivalent to a division of the network into sub-networks, sometimes partially overlapping. An informal definition says that a community is a set of nodes which are more densely connected to each other than to remaining nodes of the system [1, 2]. An exhaustive review of the methods of identification of communities has been presented in [3]. Here we note the well-known problem which arises when overlapping nodes are attached with the same strength to more than one community [4]. Another important problem is that it is difficult to indicate communities properly when communities differ significantly in their sizes [2]. Further, sometimes even though communities are well-defined, the link density is so homogeneous that it does not allow for their proper

reconstruction of the communities by used algorithms [5].

Analysis of networks can regard also local aspects of the network structure, as degree of nodes and their neighbours. In networks with some symmetry, these local properties may be the same for different nodes. This observation allows us to group nodes into classes which differentiate these local properties [6]. Large size of networks is often an obstacle in simulations. The classification of nodes allows for an effective reduction of the system size which may preserve an information of system properties [6].

The main issue of the paper is what we can infer about the class structure from the results of the analysis of the community structure.

The paper is organised as follows: At first, the analysed networks are presented together with the methods of communities and classes identification. The following Section 3 is devoted to the presentation of the results obtained from both these procedures. The last section contains our concluding remarks.

2 The analysed networks

As examples of networks which display both overlapping and symmetry, we use the networks constructed on bases of two instances of the Lindenmayer systems (L-systems) [7] – the Sierpinski triangle and the Koch curve. In the first case, apexes of triangles are identi-

fied with network's nodes (while we take into account that most of apexes belong to more than one triangle), and sides of triangles are identified with network's edges. In the case of the Koch curve, the network nodes are identified with curve intersects. In the latter case, we observe not only symmetry and overlapping communities but also differences of community sizes.

2.1 Communities

To identify communities we use the method proposed by us some years ago [8]. The idea of the method is that a link weight between two nodes increases if these nodes have a common neighbour. According to the proposed set of differential equations, the connectivity matrix evolve, and weights of some edges increase while others decrease. The evolution equation is of the form:

$$\frac{dA_{ij}}{dt} = G(A_{ij}) \sum_{k \neq i,j} (A_{ij}A_{kj} - \beta)$$

where A_{ij} - an element of the connectivity matrix, $G(x) = \Theta(x)\Theta(1 - x)$, and β is the model parameter, which specifies a threshold between meaningful and negligible values of the product of two links weights. Under these equations, the time evolution of the method is fully deterministic, and therefore free from an inherent noise of Monte Carlo methods. The evolution process leads to a subsequent division of the nodes into communities. The accepted division is the one with the highest modularity value [9, 10].

Here the connectivity matrix constructed for the analysed networks consists of 1's and 0's. If any two nodes are connected an appropriate matrix element is equal to 1, otherwise it is 0. Application of our method to such a network gives only communities which are well defined. In the case of the networks analysed here, one can expect that overlapping nodes will be identified as one-node communities. However, to be able indicate connections of those nodes with other nodes in the network we slightly disturb the connectivity matrix. Namely, we add a random number to each 0 and we subtract a random number from each 1. If the amplitude of random numbers is small enough, such a modification - for sufficiently big statistics - does not destroy the network structure. Yet, an application of the method allows to divide the network into communities which contain overlapping nodes.

As our procedure involves a disturbance of the connectivity matrix by some random number it is necessary to indicate the amplitude of the used noise. This means that now we have two parameters: just mentioned amplitude and the parameter β in the community identification procedure. The values of those parameters were found experimentally for each analysed network to obtain the division into communities which guarantees a maximal value of the modularity.

2.2 Classes

The idea of the classes is based on the symmetry observed in networks, which causes that it is possible to indicate groups of states which

are equivalent. The symmetry here means that some nodes have the same number of neighbours which belong to the same classes. This observation allows for the reduction of the system size, as the whole network can be represented in the equivalent way as the network of classes [6, 11, 12]. The procedure of classes identification takes into account not only the number of neighbours of each node, but also the weights of connections. In particular, in some cases weights of edges between two nodes in both directions (e.g. $a \rightarrow b$ and $a \leftarrow b$) may be different. Because of that the full procedure takes into account the lists of in- and out-neighbours of each node. In the case of networks analysed here, the procedure may be simplified as networks are unweighted and undirected. The noise, remarked in the preceding section, is applied only in the search of communities, and not classes. Details of the procedure of classes identification have been presented in [12]. We present here only the main features of the method. The procedure of class identification is performed as follows: At first we check degrees of the nodes of the network. Then, we assign to each node which has a given degree a unique symbol (class identifier). Subsequently we make lists of neighbours for each node, where we replace the node number with its class symbol. If the lists of neighbours for all nodes which have assigned the same symbol are not the same, we have to introduce additional distinction between nodes with the same degree. The classification is repeated until unique lists of neighbours for different class symbols are obtained.

3 Results

3.1 The Sierpinski triangle - the communities

- $N = 15$, *no noise*: In the case of the Sierpinski triangle network consisting of 15 nodes, the community identification method applied to the original connectivity matrix leads to the division of the network into 3 communities of size 3 each (community 1 - nodes 1, 2 and 4, community 2 - nodes 5, 7 and 10, and community 3 - nodes 9, 11 and 13) and 6 communities of 1 node each. The result is presented in Fig.1. For this division, the modularity is equal to 0.12 for $\beta = 0.4$. Here we see that the nodes which form one-element communities are of two types: the first type for nodes number 0, 8 and 14 and the second for nodes 3, 6 and 12.
- $N = 15$, *noise*: To indicate the connections between communities we add a noise of amplitude 0.01 to the connectivity matrix elements and we record the obtained communities for independent 1000 realizations of the noise. The maximal obtained modularity is 0.19, for $\beta = 0.15$ and the division of the network into 3 communities. The obtained result can be presented in the form of histograms of the frequency of a given node being connected to a given original community of the size different than one. Examples of the obtained histograms are presented in Fig.2. As we see, two different situations are observed for two mentioned types of nodes which originally form one node communities. For the first type (0, 8, 14), one clear maximum is observed, and for

the second type (3, 6, 12), we observe two values significantly higher than the third one. This result indicates that the nodes of the second type can be equally likely assigned to two different communities. In this case we deal with the overlapping.

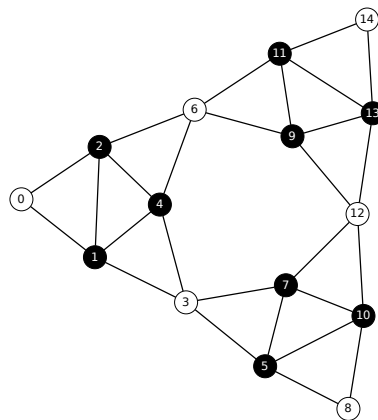
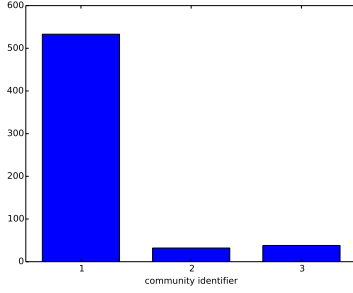
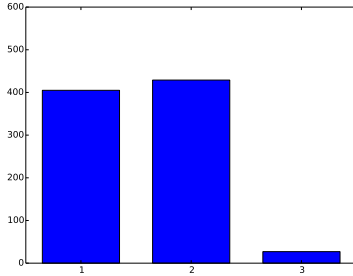


Figure 1: The Sierpinski triangle network with indication of communities for $N = 15$. Here we have 3 communities of 3 nodes each (black nodes) and 6 communities of 1 node each (white nodes)

- $N=42$, *no noise*: For a larger network of $N = 42$ we obtain 9 communities of 3 nodes each and 15 communities of 1 node each, for original connectivity matrix (see Fig.3). For this network the modularity equals 0.15 for $\beta = 0.4$.
- $N=42$, *noise*: In this case, for the connectivity matrix with noise of amplitude 0.001 and for $\beta = 0.05$, four types of nodes classified to one node communities are observed,



(a) Node number 0, analogous histograms are obtained for nodes 8 and 14 with connection to the community 2 and 3, respectively.



(b) Node number 3, analogous histograms are obtained for nodes 6 and 12 with connection to the community 1 and 3, and 2 and 3, respectively.

Figure 2: Histograms of the frequency of the connection of a given node with original communities of the size different than one for the Sierpinski triangle network for $N = 15$.

which is reflected in the histograms profiles. The optimal division is obtained for modularity equal to 0.26 and division in 9 communities. The first type form nodes number 0, 22 and 41, the second type - nodes 3, 6, 15, 29, 33 and 39, the third type nodes 8, 19, 35, and the last one – nodes 13, 20 and 27 (see Fig.4). The histograms obtained in this case allow not only to indicate the communities to which a given node is connected to, but also reveal further connections between them. A similar frequency indicate the same "distance" to a given community.

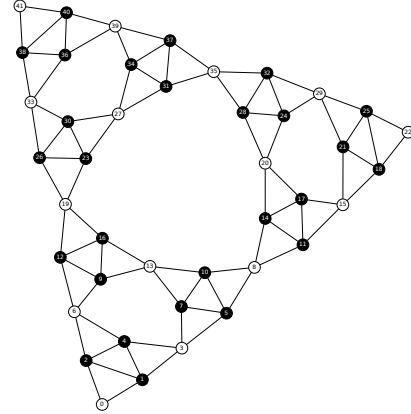
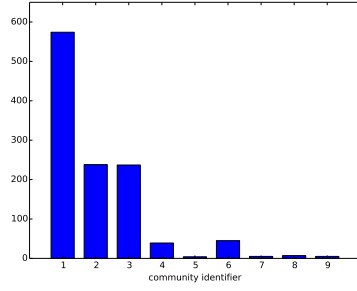
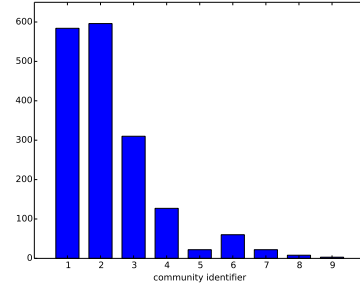


Figure 3: The Sierpinski triangle network with indication of communities for $N = 42$, here we have 9 communities of 3 nodes each (black nodes) and 15 communities of 1 node each (white nodes).

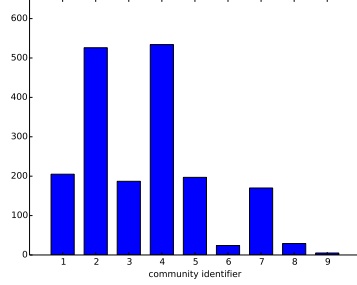
In Fig.4a we see one maximum which indicate the community with which the node 0 is directly connected to. Two subsequent bars with approximately equal height indicate the communities which are next nearest



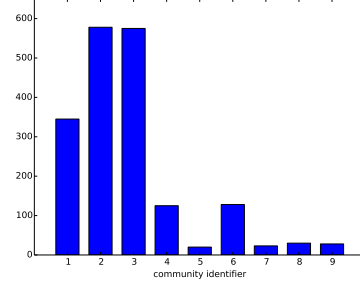
(a) Node number 0. Analogous histograms are obtained for nodes 22 and 41.



(b) Node number 3. Analogous histograms are obtained for nodes 6, 15, 29, 33 and 39.



(c) Node number 8. Analogous histograms are obtained for nodes 19 and 35.



(d) Node number 13. Analogous histograms are obtained for nodes 20 and 27.

Figure 4: Histograms of the frequency of the connections of a given node with original communities of the size different than one for the Sierpinski triangle network for $N = 42$.

neighbouring communities. In Figs.4b and 4d we see two bars approximately equal in size, which indicate nearest communities to the node number 3. The lower bar is related to the community which is connected with both just mentioned communities, and so on. Alike, in Fig.4c there are two direct neighbouring communities, and four next neighbouring communities.

3.2 The Koch curve - the communities

- *$N=116$, no noise:* In the case of the Koch curve network of size $N = 116$ the original system is divided into 24 communities of size 4 each and 20 communities of 1 node each (Fig.5). Such a division was obtained for $\beta = 0.01$ and leads to modularity equal to 0.32. The same usage of noise allows to indicate the overlapping nodes. Typical histograms obtained for this network are presented in Fig.6.

- *$N=116$, noise:* The maximal value of modularity for noisy network the modularity equals 0.42 and is obtained for noise amplitude equal to 0.001 and $\beta = 0.05$. The number of communities for those parameters equals 24. Here we observe three profiles of histograms, which reflect three types of white nodes in Fig.5. We see there two different types of nodes of degree 2 and one node type of nodes of degree 3. In the histograms it manifests in 2 or 3 bars higher than other bars (Fig.6). Also a distinction between nodes of degree 2 is observable in the histogram pro-

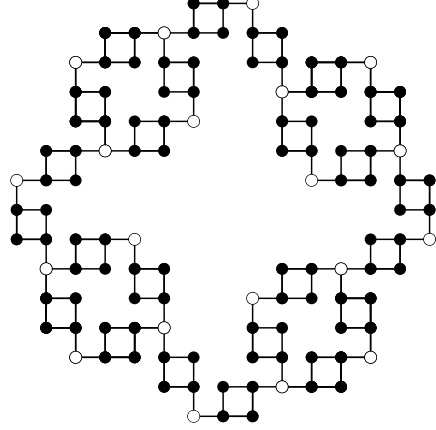
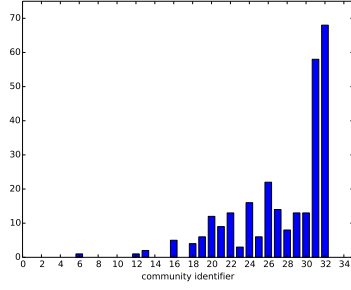


Figure 5: The Koch curve network with indication of communities for $N = 116$, here we have 24 communities of 4 nodes each (black nodes) and 20 communities of 1 node each (white nodes).

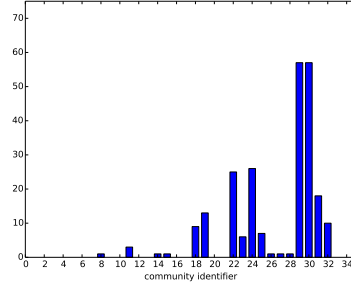
files (see Figs.6a and 6b).

- *$N=692$, no noise:* For a larger system of size $N = 692$ nodes, 24 communities of size 13, 8 communities of size 40, and 60 nodes of size 1 are obtained. The division is presented in Fig.7, where black dots indicate nodes classified to singular communities. Such a division - obtained for $\beta = 0.001$ - leads to modularity value 0.41.

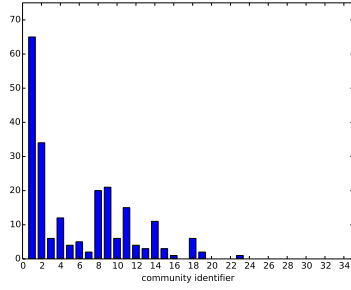
- *$N=692$, noise:* Noise amplitude 0.001 and $\beta = 0.08$ leads to the division of the network into 32 communities with modularity equal to 0.36 (220 repetitions).



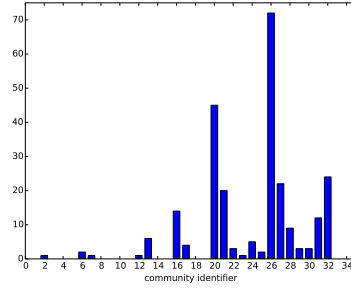
(a) First type, analogous histograms are obtained for 3 similar nodes.



(b) Second type, analogous histograms are obtained for 7 similar nodes.

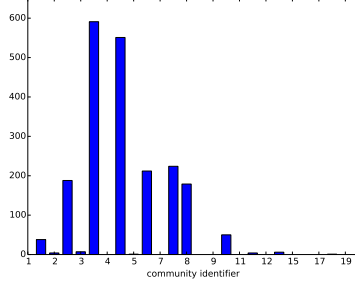


(c) Third type, analogous histograms are obtained for 15 similar nodes.

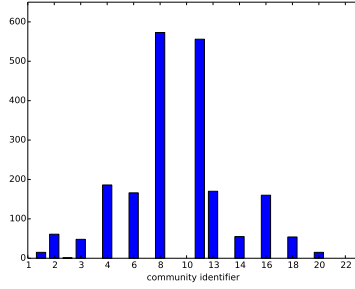


(d) Forth type, analogous histograms are obtained for 31 similar nodes.

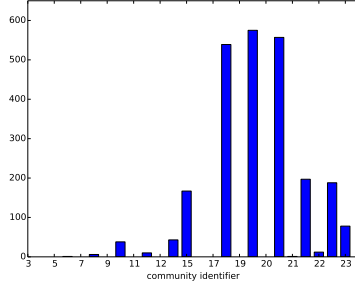
Figure 8: Histograms of the frequency of the connection of a given node with original communities of the size different than one for the Koch curve for $N = 692$.



(a) First type, analogous histograms are obtained for 7 similar nodes.



(b) Second type, analogous histograms are obtained for 3 similar nodes.



(c) Third type, analogous histograms are obtained for 7 similar nodes.

Figure 6: Histograms of the frequency of the connection of a given node with original communities of the size different than one for the Koch curve for $N = 116$.

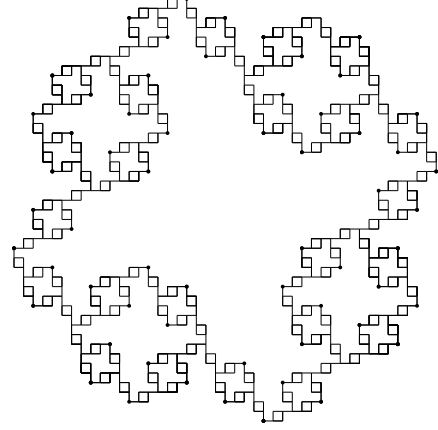


Figure 7: The Koch curve network for $N = 692$, black dots indicate nodes classified to singular communities.

<i>id</i>	<i>k</i>	<i>node's numbers</i>
A	2	0 8 14
B	4	1 2 5 10 11 13
C	4	3 4 6 7 9 12

Table 1: Classes obtained for the Sierpinski triangle network of size $N = 15$. Here nodes are classified to 3 classes (column *id*), for all but one the node degree $k = 4$.

3.3 The Sierpinski triangle - the classes

The classes obtained for the Sierpinski triangle network of size $N = 15$ are presented in Fig.9 and Tab.1. Here, nodes are classified to 3 classes, where the node degree k in one class is equal to 2 and for two remaining classes $k=4$.

In Fig.10 and Tab.2 the classes are presented obtained for the network of size $N =$

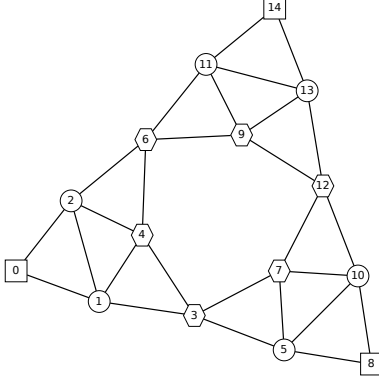


Figure 9: The Sierpinski triangle network with indication of nodes classes for $N = 15$, here nodes belong to 3 different classes indicated in the figure by \circ , \square and \diamond .

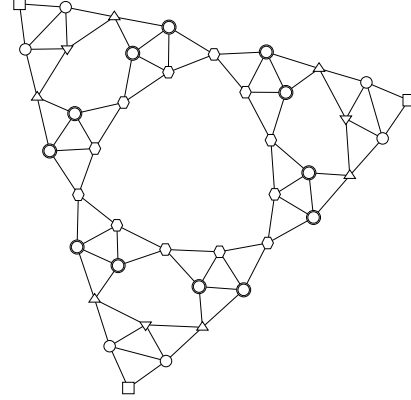


Figure 10: The Sierpinski triangle network with indication of nodes classes for $N = 42$, here nodes belong to 6 different classes indicated in the figure by six different symbols.

42. In this case 6 classes are identified, and the node degree k for all but one (for which $k = 2$) is equal to 4.

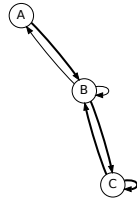
Graphs for classes, for both analysed system sizes, are presented in Fig.11. For example, for $N = 15$ there are 3 classes, as there is only three types of nodes. All nodes in class A are connected only to nodes in class B . Note that all remaining nodes, although seem topologically different, belong to the same class C . This is because each node in C has two neighbours in C and two neighbours in B .

3.4 The Koch curve - the classes

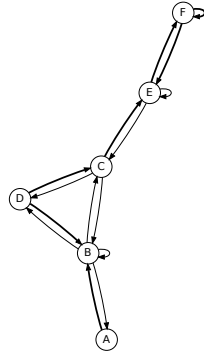
The procedure of the class identification applied for the Koch curve network indicate to

id	k	$node's\ numbers$
A	2	0 22 41
B	4	4 21 36
C	4	1 2 18 25 38 40
D	4	3 6 15 29 33 39
E	4	5 7 9 11 12 17 24 26 30 32 34 37
F	4	8 10 13 14 16 19 20 23 27 28 31 35

Table 2: Classes obtained for the Sierpinski triangle network of size $N = 42$. Here nodes are classified to 6 classes (column id), for all but one the node degree $k = 4$.



(a) $N = 15$



(b) $N = 42$

Figure 11: Graph of classes for the Sierpinski triangle network.

id	A	B	C	D	E
k	2	2	3	3	3
$\#$	12	48	24	24	8

Table 3: Classes obtained for the Koch curve network of size $N = 116$. Here nodes are classified to 5 classes (row id), in 2 classes the node degree $k=2$, and in 3 classes $k=3$, $\#$ is the number of nodes belonging to a given class.

id	A	B	C	D	E	F	G	H	I
k	2	2	2	2	2	2	3	3	3
$\#$	12	48	48	96	96	48	24	48	48
id	J	K	L	M	N	O	P	Q	
k	3	3	3	3	3	3	3	3	
$\#$	24	48	48	24	24	24	24	8	

Table 4: Classes obtained for the Koch curve network of size $N = 692$. Here nodes are classified to 17 classes (rows id), in 6 classes the node degree $k=2$, and in 11 classes $k=3$, $\#$ is the number of nodes belonging to a given class.

the presence of 5 classes for $N = 116$ (Tab.3) and 17 classes for $N = 692$ (Tab.4). In both cases node degree is equal 2 or 3.

3.5 Communities and classes

Here we ask if the number of classes can be deduced on the basis of knowledge achieved during the communities identification procedure. The analysis of histograms obtained for all nodes allows to divide nodes into

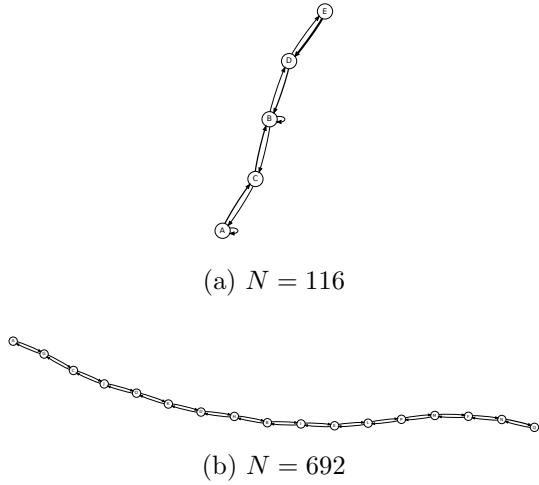


Figure 12: Graph of classes for the Koch curve network.

groups, within which the histogram profiles are similar; actually we believe that with a better statistics, they would be almost the same. This similarity means, that nodes positions in the network are analogous; for example, there are nodes which connect two other similar communities. Those similarities also underlie the class identification procedure.

Our analysis indicates that nodes which are characterized by the same histograms belong to the same class. In most cases, different histogram profile means that nodes are of different classes. However, for example in the case of the Sierpinski triangle network of $N = 42$, nodes 8, 19, 35 and 13, 20, 27 have different histogram profiles, but they belong to the same class.

The analysis of histogram profiles involves

only nodes which in the absence of noise form one-node communities, and do not take into account other nodes of the system. In some cases they are classified to the classes which cover overlapping nodes, but sometimes they form new classes. For small and simple systems, the histogram profiles may be also used to predict the number of types of nodes classified to original communities (with no noise). For each node of the same histogram profile one can indicate its nearest neighbours. Then, existence of an intersection between lists of neighbours obtained for similar nodes can be checked. Non-zero size of such an intersection means that nodes of different types have a common neighbour. And vice versa, if there is no intersection – nodes are not connected by such a neighborhood. Such information allows us to indicate the number of types of nodes which form communities in the case without noise.

In the case of the Sierpinski triangle of size $N = 15$ and 42 , as it was already written, 2 and 4 histogram profiles, respectively, are obtained. We then construct lists each of which contains nearest neighbours of nodes for which the same profile is observed. Next, intersection of obtained lists was found. As a result we obtain that in the case of the smaller system the lists contain common nodes, while for the larger system intersection of some of lists is an empty set. It appears, that in the first case nodes are of the same type, while in the second they can be divided into two categories (see Tab.5).

In the case of the Koch curve network of size $N = 116$, the proper analysis of mentioned intersections involve taking into ac-

	a	b	c	d
a	x	x	-	-
b	x	x	x	x
c	-	x	x	x
d	-	x	x	x

Table 5: Intersection existence between lists of nearest neighbours of nodes for which given histogram profile (a , b , c or d) for the Sierpinski triangle network of $N = 42$ – sign 'x' indicate not empty intersection, and '-' empty intersection.

	a	b	c
a	x	-	x
b	-	x	x
c	x	x	x

Table 6: Intersection existence between lists of nearest and next nearest neighbours of nodes for which given histogram profile (a , b or c) for the Koch curve network of $N = 116$ – sign 'x' indicate not empty intersection, and '-' empty intersection.

count also next nearest neighbours. The result obtained both for the Sierpinski triangle network and the Koch curve network are the same as the results obtained from the class identification procedure. For larger systems, or in the case of larger communities such analysis is not sufficient to indicate of nodes classes, as in this case connections schemes of the nodes inside communities are more complicated.

4 Discussion

In this paper we present a new application of the formerly proposed system of differential equations [8] to identify the structure of overlapping communities. The method is applied twice, for unnoised and noised initial values of the coefficients of connectivity matrix. In the first case, part of nodes appear to form one-node communities. In the second case, these nodes are attached to other communities with different probabilities. The distribution of this probability is found to be a useful characteristics of a given node.

Next, the class structure of the investigated fractals is determined, according to the method developed in [6, 11]. Having this done, we concentrate on the nodes which have been found to fall into one-node communities. The result is that once two such nodes are described with the same histograms, they fall into the same class.

This procedure may be used for any

<i>network</i>	<i>N</i>	<i>no noise</i>		<i>with noise</i>			# <i>classes</i>
		# <i>comm.</i>	<i>m</i>	# <i>comm.</i>	<i>m</i>	# <i>hist.</i>	
<i>Sierpinski</i>	15	3(3)+6(1)	0.12	3	0.19	2(1)	3
<i>triangle</i>	42	9(3)+15(1)	0.15	9	0.26	4(2)	6
<i>Koch</i>	116	24(4)+20(1)	0.32	24	0.42	3(2)	5
<i>curve</i>	692	24(13)+8(40)+60(1)	0.41	32	0.36	4(n.a.)	17

Table 7: Results summary. Abbreviations used in table: *N* - network size, # *comm.* - number of communities, *m*- modularity, # *hist.* - number of histogram profiles.

network, in particular other fractals.

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References

- [1] Newman M.E.J., "Modularity and community structure in networks", Proc. Natl. Acad. Sci. U.S.A. 103 (2006) 8577
- [2] Fortunato S. and Barthelemy M., "Resolution limit in community detection", Proc. Natl. Acad. Sci. U.S.A. 104 (2007) 3641
- [3] Fortunato S., "Community detection in graphs", Physics Reports 486 (2010) 75
- [4]
- [5] Radicchi F., "A paradox in community detection", arXiv:1312.4224
- [6] Krawczyk M.J., "Topology of space of periodic ground states in antiferromagnetic Ising and Potts models in selected spatial structures", Physics Letters A 374(2010) 2510
- [7] Prusinkiewicz P., Lindenmayer A., "The Algorithmic Beauty of Plants (The Virtual Laboratory)", Springer Science+Business Media, 1990
- [8] Krawczyk M.J., "Differential equations as a tool for community identification", Phys. Rev. E 77 (2008) 065701(R)
- [9] Newman M.E.J., Girvan M., "", Phys. Rev. E 69 (2004) 026113
- [10] Newman M.E.J., "Modularity and community structure in networks", Proc. Natl. Acad. Sci. U.S.A. 103(23) (2006) 8577
- [11] Krawczyk M.J., "Symmetry induced compression of discrete phase space", Physica A 390, 2181 (2011)
- [12] Krawczyk M.J., "Classes of states of discrete systems", submitted